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Using Geometry to Teach and Learn Linear Algebra

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ABSTRACT. Linear algebra is a difficult topic for undergraduate students. In France, the focus of beginning linear algebra courses is the study of abstract vector spaces, with or without an inner product, rather than matrix operations as is common in many other countries. This paper presents a study of the possible uses of geometry and "geometrical intuition" in the teaching and learning of linear algebra. Fischbein's work on intuition in science and mathematics is used to analyze the treatment and use of geometry in linear algebra textbooks as well as mathematicians' and students' uses of geometry in linear algebra. I indicate the possibilities and limitations of such uses of geometry and make suggestions for a linear algebra course that uses geometry to support learning.

1. Introduction

Prior to a description of the issues studied herein, it is necessary to clarify the use of the expression "linear algebra" in this paper. "Linear algebra" is used in accordance with the teaching context in France. This may be different from many other countries, where undergraduate students encounter mostly matrix-oriented courses. Though the issues I study are not specific to France, a brief description of the historical background of the teaching of linear algebra in France can help to clarify the context of the study.

Linear algebra was first taught in French universities at the graduate level, in 1939. The first courses were strongly connected with the study of Hilbert spaces. During the 1960s, the introduction of linear algebra into the secondary school curriculum led to many discussions among French mathematicians. These discussions included some contention about the presentation of geometry.

French mathematicians took two main opposing views. The first one, discussed in detail in Choquet's book *Teaching Geometry* (1964), recommended presenting geometry defined by axioms (independent of linear algebra), then using it for an intuitive presentation of linear algebra. Mathematicians like Dieudonné took a different view. They preferred to start directly with linear algebra because, as he said, geometry was a mere application of linear algebra. Understanding geometry was an immediate consequence of understanding linear algebra. During the "modern mathematics" reform, Dieudonné's position was adopted in France's national curriculum, and linear algebra started to be taught in secondary school. However, difficulties encountered by students led to the failure of this approach, and linear algebra disappeared from the secondary school curriculum during the 1980s. Since 1986, linear algebra – banished from secondary education – has been a requirement for undergraduate science students. It remains a difficult topic for them.

Many mathematicians have claimed that using geometry or "geometrical intuition" helps students in their understanding of linear algebra. This claim raises several questions. In the French teaching context, "linear algebra" is clearly identified as linear algebra in abstract spaces, with or without an inner product. I will refer to this later as "general linear algebra" or "the general theory." But what is meant by "geometry"? It can be geometry taught in secondary school[1] or Euclidean geometry (in its historic, axiomatic meaning). For some mathematicians, linear algebra itself (or at least parts of it) are a kind of geometry. Another question is: What is meant by "geometrical intuition"? It is certainly linked with the possibility of using drawings or mental pictures. But intuition does not mean only visualization, and there is no doubt that it has other aspects. Determining how geometrical intuition can help students in their learning of linear algebra, and whether mathematicians try to develop geometrical intuition in their linear algebra courses by specific choices, are additional issues.

Several published studies have investigated how geometry or geometrical aspects of linear algebra can be used to introduce the general theory. They have reported that teaching based on a geometrical approach can improve students' understanding, but have pointed out difficulties stemming from such a choice (see §2). These studies have confirmed that possible interventions of "geometrical intuition" in linear algebra requires a thorough study. This was the aim of my doctoral dissertation, which contains most of the results presented here. This paper has three main parts: a grounding of the study in related theory and research, analysis of the uses of drawings and geometry in teaching linear algebra, and particular results from interviews about the role of $\mathbb{R}^2$ and $\mathbb{R}^3$ in linear algebra teaching and learning.

Overall presentation of the study (§2). I start with the main theoretical framework for my research. Studying the question of intuition requires appropriate tools; I found them in Fischbein's (1987, 1993) work on intuition in science and mathematics. In this section, I present the notion of intuitive models, along with the research questions it allowed me to formulate. I also describe the setting of my study within the context of related studies. Though these constituted a starting point for my study, I present them at the end of the section in order to interpret their results in terms of the notion of geometrical intuition.

Geometry in linear algebra courses (§3). In this section, I present results about mathematicians' uses of geometry and drawings in linear algebra courses. These results come from the analysis of mathematicians' responses to a questionnaire. Since my research took place in France, the results of the questionnaire are influenced by French teaching of linear algebra and geometry. However, this is not the case for the results in other sections.

Linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ as a source of geometrical intuition (§4 and §5). In these two sections, I focus on a particular intuitive model: linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ with the dot product. I first establish some possibilities and limitations of that model through a textbook study (§4.) I then present a detailed analysis of the effects of the use of that model among students solving a problem in $\mathbb{R}^n$ (§5).

[1]In France, the geometry taught in secondary school is mostly plane and space vector geometry. Vectors are defined intuitively; there are no axioms presented.

2. Theoretical Framework and Research Questions

Many mathematicians have attempted to clarify the nature of intuition in mathematics, in students' learning as well as in their own research. Most of them have distinguished different kinds of intuition taking a psychological approach. But the writings of few provide a means for a precise analysis of intuition. Fischbein (1987) made a careful study of the nature of intuition, its function, and the factors that can shape it. His work is therefore the basis of my theoretical framework. Below I present those elements of Fischbein's work that are relevant for the present study.

2.1. Intuition in Mathematics: Fischbein's Theory.

2.1.1. *Intuition and the need for certitude.* According to Fischbein, every human being needs to act in accordance with a credible reality. Even within a conceptual structure, one's reasoning endeavors need a form of certitude. The role of intuition is to provide that kind of certitude. For Fischbein, intuition is synonymous with intuitive knowledge. It is a type of cognition, characterized by self evidence, immediacy, and certitude; it always exceeds the given facts. Productive reasoning requires intuitively acceptable cognition. Thus, when a notion is, for a given person, intuitively unacceptable, that person will probably produce (deliberately or unconsciously) a more acceptable substitute. Such a substitute is called by Fischbein an intuitive model. Models are a central factor of intuition in mathematics. A large part of Fischbein's book (1987) is devoted to them. In the following section, I present those aspects of the models used in my study.

2.1.2. *Intuition and models.* Also, according to Fischbein:

> A system B represents a model of a system A if, on the basis of a certain isomorphism, a description or a solution produced in terms of A may be reflected consistently in terms of B and vice versa. (1987, p. 121)

This definition is very general; the word "system" used in it can have several meanings. The following examples (all of them related to the present study), will be useful to make the definition precise.

Example 1. A "system" can be restricted to a single notion. For example, the system A can be the notion of vector in the plane, and the model B the drawing of arrows on a sheet of paper.

Example 2. A system can also be a whole theory: complex numbers (system A), associated with the vector geometry of the plane (system B).

Example 3. A system is not always a conceptual system: physical space (system A) can be associated with $\mathbb{R}^3$ considered as a vector space with an inner product (system B).

By Fischbein's definition, a property in B may be "reflected consistently" in A. This means that the property can somehow be translated from one system to the other. Let us consider in Example 1 the relation R_A: "the vector u is the sum of v and w." R_A can be associated with the drawing R_B of a parallelogram, whose sides are the arrows associated with v and w, and one of whose diagonals is the arrow associated with u. The relation R_A in A corresponds to a consistent relation R_B in B.

But a model can also lead to misconceptions if it is wrongly used. In Example 1 again, students sometimes claim that two vectors in the plane have the same direction because the associated arrows are both pointing "up, on the right." In this case, the word "direction" exists in both systems. But the notion of the "direction" of a vector in the plane cannot be reflected consistently in the common notion of "direction" of an arrow.

In most cases, the word "isomorphism" used by Fischbein in his definition is not a mathematical isomorphism (but it can happen, as in Example 2). Rather, the word "isomorphism" is used to indicate a particular set of relations between some objects and properties of A and some objects and properties of B. Extending this to additional relations is likely to be misleading.

The three examples given above correspond to three different kinds of models. Among the models distinguished by Fischbein are the following: figural models, abstract and intuitive models, and analogical and paradigmatic models. I discuss each in turn.

Figural models. Fischbein distinguishes between intramathematical and extramathematical models. In the case of an intramathematical analogy, the original and the model are both mathematical theories or objects. In contrast, an extramathematical object is something that does not lie strictly within mathematics. It is not a mathematical object, collection of objects, or theory. The extramathematical models that I will study here correspond to the use of drawings. I refer to such models as figural models. Here "drawings" means pictorial representations.

A figural model can be related to geometrical notions; for example, the calculation of the distance from a given point p to a given plane F in 3-space can be associated with a drawing of a parallelogram, a point, and a dotted line containing the point, perpendicular to the plane (Figure 1). But the same drawing can also be associated with a polynomial problem: calculation of the distance from x^4 to the set F of polynomials of degree less than 3, in the space $\mathbb{R}_4[X]$ with the inner product defined by $< p|q >= \int_0^1 p(x)q(x)dx$.

Three-dimensional objects, and computer-generated graphics are other kinds of extramathematical models that can be used in linear algebra. I do not discuss them here because they are used only sparsely in France.

Abstract and intuitive models. Models fall into two distinct categories: abstract and intuitive. Some mathematical objects are abstract models for concrete realities. In Example 3, $\mathbb{R}^3$ with the dot product is an abstract model of physical space.

In contrast, an intuitive model is one that seems concrete to the perceiver. Figural models are obviously intuitive. But a mathematical object can also be an intuitive model for someone who perceives it as a reality. In Example 2, vector geometry in the plane is a model for complex numbers. The existence of this intuitive model was very important in the emergence of the notion of complex numbers because it legitimated their existence. Complex numbers gained legitimate status with the work of Gauss (1831), who presented a geometric interpretation of imaginary quantities. Though other mathematicians like Wessel(1799), Buée (1805), and Warren (1828) had proposed such interpretations before, they probably lacked Gauss' influence on the mathematics community.

Analogical and paradigmatic sub-categories. Intuitive models themselves have two sub-categories: analogical and paradigmatic. An analogical model is external to the original object modeled; the model and the original belong to two distinct systems. An analogical model provides the reasoning process with a source of research hypotheses. According to Fischbein, two systems are said to be analogical if a partial similarity exists, that can lead a person to assume additional similarities.

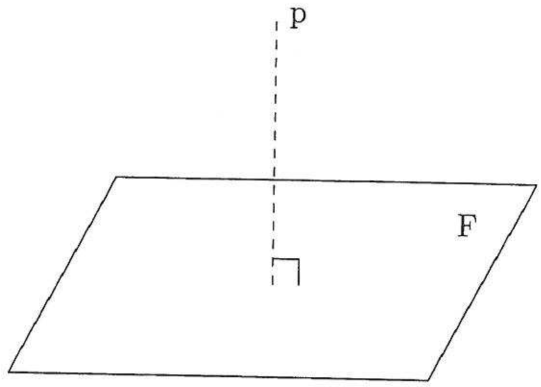


FIGURE 1. Distance from a point (or a polynomial) to a plane (or a subspace of polynomials).

Analogy justifies plausible inferences. Analogies become models if they can be productively used in reasoning. Figural models associated with mathematical models are analogical models.

A paradigmatic model is a subclass of the original that is used as a model of the original. It is a particular exemplar of the original. That is, Fischbein's concept of paradigmatic model is similar to the concept of prototype in cognitive semantics. One's understanding of the original system is influenced (correctly or incorrectly) by the paradigmatic model. The original mathematical object is represented in one's reasoning by that exemplar, and not by its abstract definition. For example, using $\mathbb{R}^3$ as a model for the concept of vector space can lead to attributing properties of $\mathbb{R}^3$ to all vector spaces. Someone using that model might claim that two two-dimensional subspaces can never be supplementary subspaces.

The above theoretical considerations led me to a definition of "geometrical intuition." It also provided me with an appropriate framework for formulating and investigating my research questions in a consistent way.

2.2. Research Questions. The first stage here is to clarify what can be called a geometry. Some mathematicians may consider linear algebra to be a geometry; this cannot be relevant in our case. Because my analysis mostly takes place in a teaching context, I am naturally referring to geometry as a mathematical domain. The definition of intuition given by Fischbein emphasizes credible reality. In the use of geometry in the teaching and learning of linear algebra, the link with reality is central. For this reason, I call "geometry" a mathematical theory whose main purpose is to provide an abstract model (using Fischbein's terminology) for physical space; it is notably restricted to three dimensions. The geometry taught at school is such an abstract model, as is axiomatic Euclidean geometry.

I will term "geometric intuition" the use of models stemming from a geometry. These models are intuitive models; they can be either analogical or paradigmatic. In the first case, this constitutes an intramathematical analogy.

Because of the nature of geometry, a geometric model will always be associated with an extramathematical, figural model. A geometric model can thus smuggle uncontrolled elements into the reasoning process. For example, when studying the general notion of quadratic form, students encounter self-orthogonal vectors.

That situation cannot be associated with anything in two-dimensional Euclidean geometry; it is contradictory to a drawing representing two orthogonal vectors in the plane.

In the following study, I will use geometric and figural models rather than the general expression "geometric intuition." The research questions can be formulated as follows:

(1) What are the possible uses of geometric models in linear algebra?
(2) How do mathematicians and students use geometric and figural models in linear algebra?
(3) What are the consequences of the observed uses of models on students' practices and thinking processes?

These are very general questions, for which my work only provides partial answers. However, some previous related research on linear algebra provides hints about the use of geometric and figural models, even if it was not formulated in those terms. I discuss this research in the following subsection, together with the setting of my study in the context of these works.

2.3. Related Research. Much research on linear algebra addresses the question of possible uses of geometry or geometric aspects of linear algebra. Here I only present research in which that question is central and whose results are meaningful in terms of intuitive models. I also discuss its connections to and differences with my own work.

2.3.1. *Modes of description.* Hillel (2000) identified three modes of description in linear algebra: the *abstract mode*, the *algebraic mode*, and the *geometric mode*. The abstract mode uses the language and concepts of the general theory (e.g., vector space, dimension, kernels). The algebraic mode uses the language and concepts of the theory in $\mathbb{R}^n$ (e.g., matrices, systems of equations). The geometric mode uses the language and concepts of 2- and 3-space (e.g., points, lines, planes). His approach was quite different from mine because Hillel studied the three modes and the mechanisms that enable one to move from one mode to another, but he did not examine the question of geometric intuition.

Nonetheless, his description of students' difficulties with the geometric mode can be interpreted as consequences of an irrelevant use, or of the use of irrelevant components, of a figural model. Difficulties attached to the point and arrow depictions of a vector are the most striking. Hillel observed that most mathematicians use both depictions and he described wrong interpretations of some representations by students. For example, it is well known that students may claim that it is possible for two 1-dimensional subspaces to have an empty intersection. This phenomenon can be interpreted as a misleading intervention of a figural model: 1-subspaces are represented as straight lines, and students believe that they can be parallel. In this case, the representation of a straight line that does not contain the origin is an irrelevant component of the model.

The question of how point and arrow depictions in linear algebra affect students' understanding could be studied from the point of view of geometric intuition. Although I encountered related difficulties in my work with mathematicians and students, I will not address them explicitly.

2.3.2. *Cabri-Geometry in linear algebra.* Sierpinska, Dreyfus, and Hillel (1999) designed a learning environment with Cabri-Geometre II software for the notions of vector space, linear transformation, and eigenvector. That environment was explicitly intended to use geometric intuitions. The intuitive model was provided by Cabri vectors. I focus here on the difficulties of students discussed by the authors. At one stage in the reported activities, students encountered the task: "Find the coordinates of a vector v in the basis v_1, v_2." The vectors v, v_1, and v_2 were Cabri-vectors constructed on the screen by the students. They immediately reorganized their construction to obtain orthogonal vectors v_1 and v_2. They went on doing a mechanical calculation of coordinates, not attending to the initial meaning of the problem. That phenomenon can be interpreted in terms of intuitive models. The notion of "coordinates" is strongly associated with the drawing of two orthogonal axes. The students referred to that model because they did not have an appropriate figural model for the notion of basis vectors at their disposal. The authors had intended to avoid an explicit introduction of the notion of basis and expected the students to develop an intuition of it. It is interesting to observe here the emergence of a very familiar model, one that has the appearance of credible reality. It created a misleading geometric intuition.

Similar difficulties were observed by Sierpinska (2000). In a further analysis of the same teaching environment, she identified a phenomenon that she described as: "Thinking of mathematical concepts in terms of their prototypical examples rather than definitions." For example, some students, when asked to construct a linear transformation with given values on a basis, looked for a well-known geometric transformation (dilation, rotation, etc.) or for linear combinations of these transformations. These students were using a geometric model for linear applications: the model of well-known transformations of the plane. That model may have been derived from previous courses,[2] but it was insufficient for the given task. In this case as in the previous one, a familiar, misleading figural model emerged.

To have the appearance of credible reality, an intuitive model must be very familiar to students. The construction of a model (as a cognitive object) is a long process that requires regular and frequent rehearsal of the elements of the model. Teaching designed to help students form intuitive models must thus be long-term, regular teaching; otherwise more familiar models are likely to emerge.

2.3.3. *The concreteness principle and geometric models.* The *concreteness principle* was stated by Harel (2000) as follows:

> For students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the students' eyes; that is to say, the student has mental procedures that can take these objects as inputs (p.177).

The definition of model given by Fischbein applies to the models mentioned here because there exists an isomorphism between a subclass of the model and a subclass of the corresponding mathematical structure. The concreteness principle is quite close to the following assumption, formulated in Fischbein's terminology: "For students to abstract a mathematical structure from a given model of that structure, that model must be an intuitive model for the student." However, establishing whether "conceptual entities" and "intuitive models" are equivalent would require a specific study.

[2]In France, the same kind of answers can be produced by students using transformations they studied in secondary school geometry.

Harel (2000) conducted a linear algebra teaching experiment using his concreteness principle and a geometric, paradigmatic model. That model can be considered as paradigmatic because it consisted of a geometric presentation of vector spaces; it was associated with a figural model. The studied teaching experiment had positive effects on the students' performances in linear algebra, including their ability to prove general linear algebra results. Students belonging to the group (Group B) that followed the experimental teaching seemed to have had a better control of the correctness of their answers than the students (Group A) who were taught linear algebra without geometric representations. The reason for that observation could be that the students of Group B formed an intuitive model that helped them to check the consistency of their reasoning. But Harel also observed difficulties attached to the use of a geometric model in linear algebra teaching. Some students could be captured by the model, and stay inside of it, instead of moving up to the general theory.

This brief review of previous research shows that several results relative to geometrical intuition in linear algebra have already been established. Namely, the use of a figural model can lead to inappropriate intuitions and a geometric model can be an obstacle if students stay captured in it. An intuitive model must consist of very familiar objects in order to have the appearance of credible reality.

Before starting with the study of mathematicians' choices and the presentation of my results, I need to mention an essential difference between the research presented above and my own study. Their authors elaborated and discussed linear algebra teaching experiments. I will consider the possibilities of elaborating teaching using geometric models, but my study deals with ordinary linear algebra courses taught at university.

3. Mathematicians and Geometric Models in Linear Algebra

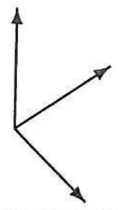
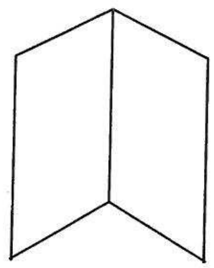
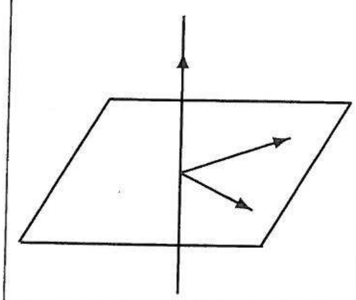
Having observed the difficulties encountered by students in their learning of linear algebra, many mathematicians have recommended that geometry be taught before the general theory. But the content of such a geometry course, and the way it might be used to learn linear algebra, depends on mathematicians' views, and those can be very different from one teacher to another. In order to have a better idea of the different uses of geometrical (and associated figural) models by mathematicians in their linear algebra courses, I created a questionnaire for mathematicians (see Appendix A).

I first present my analysis of the mathematicians' answers to the part of the questionnaire devoted to the use of drawings in linear algebra, then I present the conclusions of the analysis of the entire questionnaire. The questionnaire was given to mathematicians who answered it outside of my presence. They took about one hour to complete the entire questionnaire. I collected 31 questionnaires, completed by mathematicians of various ages and research subjects, all of them having recently taught linear algebra.

3.1. Mathematicians' use of drawings in linear algebra. The two questions relevant to mathematicians' use of drawings in linear algebra in the questionnaire were:

Question 3.1: For each of the drawings in Table 1, indicate if you use it in your linear algebra courses; if the answer is "yes" indicate which notions

TABLE 1. Question 3.1

Drawing	Used (yes/no)	That drawing illustrates
		
		
		

or properties you illustrate with it. (You can mention several uses of the same drawing.)

Question 3.2: If you use other drawings, draw them in the following table, and indicate the interpretation(s) you associate with them. (A blank table with five lines followed.)

Twenty-eight mathematicians answered Questions 3.1 and 3.2. For the analysis of the responses, I used the following criteria:

- Are the drawings in Question 3.1 used by the teacher?
- Does he (or she) mention other drawings (Question 3.2)?
- Does he (or she) mention interpretations of the drawings related to a general vector space or limited to dimension 3?

My analysis led to the following conclusions. In general, these mathematicians did not use many drawings in their linear algebra courses. Of the 28 responding, 16 mathematicians mentioned other drawings they might use in their courses. The average number of drawings mentioned by the 16 mathematicians was 2.25; this is very low considering the fact that there were five lines that could have been filled in the blank table given in the questionnaire. The average number of drawings per teacher, for both parts of the question, was only 3.2.

Moreover, most of the drawings were reported as being used to illustrate situations occuring in $\mathbb{R}^2$ or $\mathbb{R}^3$. Only 12 (43%) of the mathematicians proposed additional interpretations referring to an abstract vector space, rather than to $\mathbb{R}^2$ or $\mathbb{R}^3$. For example, for the first drawing proposed in Table 1, nine mathematicians gave the interpretation: "basis of the space,"[3] and three "orthogonal basis of the space," while only three of them mentioned the general notion "orthogonal basis," and only one the general notion of basis. For the second drawing, eight mathematicians mentioned using it in teaching as an intersection of planes, and five as an intersection of subspaces.

The drawings volunteered by the mathematicians were not very different from those provided in the questionnaire (Table 1). Except for two quadric surfaces, what the mathematicians offered were mostly combinations of planes, lines (plain or dotted) and vectors. Only five of them were drawings in the plane; the 31 others were perspective drawings evoking 3-space, even if they were used to illustrate situations in a general vector space. Space may have seemed more representative than the plane, a better candidate for a paradigmatic model. The notions illustrated included projections (3), orthogonal projections (4), symmetries (2), rotations (2), supplementary subspaces (3), and coordinates of a vector (2).

In fact, most of the notions and properties mentioned by the mathematicians already would have been encountered by students in secondary school geometry in France: lines, planes, symmetries, and projections. This was not the case for the few examples reported about supplementary subspaces and rotations around an axis. The three drawings proposed in Table 1 are used in secondary school textbooks. The first and the third occur frequently in the space geometry course; the second is used to illustrate the intersection of two planes in space, and sometimes the corresponding system of equations. So, the second drawing in Table 1 would be the least familiar of the three for secondary school students; it was also the least mentioned by the mathematicians. For the second drawing, 15 (54%) of the mathematicians declared that they used it in their linear algebra courses, whereas 23 (82%) used the first drawing and 20 (71%) used the third. These mathematicians did not have a well developed, specific, use of drawings for linear algebra. They reported drawing them mostly when presenting examples in a geometrical context.

3.2. Analysis and Discussion of the Mathematicians' Responses. Considering the responses to the entire questionnaire, I was led to distinguish three groups of mathematicians, including 24 of the 26 mathematicians who answered all the questions.[4]

Group A: Many drawings, geometry presented after linear algebra. There were only four mathematicians in Group *A*. These mathematicians used many drawings in their linear algebra courses. The figural model corresponding to their reported use of drawings was associated with a part of linear algebra that was to be used as

[3]The term "space" refers here directly to geometry. In French, the word "space" used alone means "geometrical 3-space."

[4]I used statistical tools for that global analysis, but the small number of questionnaires prevented me from referring to the statistical results without explicitly examining the effective content of the questionnaire. So, the statistical results were only a way to identify possible connections and the conclusions result from a direct observation of the questionnaires.

a paradigm for the whole theory: the study of $\mathbb{R}^2$ and $\mathbb{R}^3$. Such use of drawings would be a geometric model, according to the definition of that concept I use here. Their use of a paradigmatic model can also be a wider part of linear algebra, not limited to dimension 3, and could refer to a general vector space.

Group B: Almost no drawings, geometry presented after linear algebra. The eight mathematicians in Group B used no, or only a few, drawings. They did not refer to a geometric model to introduce linear algebra. They prefered to introduce the general theory directly and to present geometry afterwards as an application thereof.

Group C: Many drawings, geometry presented before linear algebra. The 12 mathematicians in Group C referred to an analogical geometric model, stemming from a geometry independent of linear algebra. They used many drawings in the geometry course and also in linear algebra. The drawings involved were more or less the same in both cases.

The two last tendencies, of Groups B and C, were very close to positions observed in France during the reform of modern mathematics. The structural choice of Group B corresponds, more or less, to Dieudonné's views. In contrast, Choquet advocated a presentation of geometry preceding linear algebra like the mathematicians of Group C. Groups B and C included 20 of the 26 mathematicians who completed the entire questionnaire. This may have been an indication of the influence, still very strong, of the discussions held before and during the reform of modern mathematics on the choices of French mathematicians in their linear algebra courses.

Only the mathematicians of Group A seemed to have escaped that influence. They proposed to students a geometric model *inside* linear algebra (and thus paradigmatic). That choice deserves a special study. I will now focus on this choice, and more precisely, on the use of linear algebra in $\mathbb{R}^2$ and $\mathbb{R}^3$ as a geometric model for general linear algebra.

4. The $\mathbb{R}^2$-$\mathbb{R}^3$ Model

The study of $\mathbb{R}^2$ and $\mathbb{R}^3$ as vector spaces with an inner product is a geometry. According to the definition given in Section 2, this is an abstract model for physical space. It seems to be a good candidate for a geometric model in the teaching of inner product spaces. I will call it "the $\mathbb{R}^2$-$\mathbb{R}^3$ model." I start this section with an overview of its possibilities and limitations. Then I present an example of the use of that model in a textbook.

4.1. Possibilities and limitations of the $\mathbb{R}^2$-$\mathbb{R}^3$ model. The $\mathbb{R}^2$-$\mathbb{R}^3$ model can be associated with a figural model, and coordinates offer a natural way to introduce $\mathbb{R}^n$. The link between $\mathbb{R}^n$ and other n-dimensional inner product spaces is evident for mathematicians.

Historical analysis (Dorier, 2000) of the development of linear algebra has suggested that axiomatic linear algebra finally emerged after several works about infinite dimensional spaces, as a way of unifying different mathematical domains. Linear algebra is a general theory designed to unify several branches of mathematics. Presenting linear algebra concepts only in $\mathbb{R}^2$ and $\mathbb{R}^3$ can appear very arbitrary to the students (see, for example, Robert's (1998) work about generalizing, unifying, and formalistic notions and Harel's (2000) Necessity Principle). When limited to $\mathbb{R}^2$ and $\mathbb{R}^3$, some concepts and properties of linear algebra may only seem to be geometrical tautologies to students. Below are some examples.

- The definition in $\mathbb{R}^2$ and $\mathbb{R}^3$ of a basis as a family of vectors that are linearly independent and spanning the whole space cannot appear as necessary to students. The notion of dimension is implicit and self-evident in that context; therefore a basis of $\mathbb{R}^2$ (or $\mathbb{R}^3$) is defined as a set of two (or three) linearly independent vectors. The notion of spanning the whole space does not seems to be required in that context.
- The property of existence of a basis for a given space is fundamental in general linear algebra. In $\mathbb{R}^2$ and $\mathbb{R}^3$, it appears to students as a observable fact. More generally, results stating the existence of a mathematical object are only needed in a theoretical context, where that existence cannot be directly observed.

Yet, there exist concepts and properties already relevant in $\mathbb{R}^2$ and $\mathbb{R}^3$ that can be generalized to any vector space (in fact, most of these properties occur in spaces with an inner product). For example, the Pythagorean Theorem, which is presented in the plane in secondary school and used then in several exercises, can be generalized to any space with an inner product: for a set $\{e_1, ..., e_k\}$ of orthogonal vectors, $|e_1 + ... + e_k|^2 = |e_1|^2 + ... + |e_k|^2$.

But there are obviously limitations to the use of the $\mathbb{R}^2$-$\mathbb{R}^3$ model. Some notions and properties of general linear algebra are not relevant in that context. And, the possibility of unification, central in linear algebra, is lacking in such a presentation. Moreover, the generalization to $\mathbb{R}^n$ and then to other vector spaces may not be natural for all students. For mathematicians, $\mathbb{R}^n$ is a natural model for any other real vector space of dimension n because of the structural isomorphism. For students, considering a polynomial or a function as a vector is the result of a long process.[5] The use of drawings may aid students' understanding even when the vector space is different from $\mathbb{R}^2$ or $\mathbb{R}^3$.

I now will now make these general considerations precise by examining a university textbook that uses the $\mathbb{R}^2$-$\mathbb{R}^3$ model.

4.2. A textbook using the $\mathbb{R}^2$-$\mathbb{R}^3$ model. *Linear Algebra Through Geometry* is a textbook by Banchoff and Wermer (1991), designed for undergraduate students. The title clearly announces that the authors intend to use geometry to introduce and illustrate linear algebra. In the book's preface, the authors say:

> In this book we lead the student to an understanding of elementary linear algebra by emphasizing the geometrical significance of the subject. Our experience in teaching undergraduates over the years has convinced us that students learn the new ideas of linear algebra best when these ideas are grounded in the familiar geometry of two and three dimensions. Many important notions of linear algebra already occur in these dimensions in a non-trivial way, and a student with a confident grasp of the ideas will encounter little difficulty in extending them to higher dimensions and more abstract systems (Banchoff & Wermer, 1992).

[5]It is certainly linked with encapsulation; considering polynomials or functions as vectors means, in particular, considering them as objects instead of processes (Dubinsky, 1991).

The approach of the authors is clearly stated here: first build a geometric model limited to dimensions 2 and 3. They claim that the model will help students when learning linear algebra because these students will only have to extend now familiar notions.

Analysis of the book, which I do not give here in detail, made clear that the geometric model proposed by the authors is the $\mathbb{R}^2$-$\mathbb{R}^3$ model. Chapters 1, 2 and 3 are dedicated to it. Chapter 4 is a transition; it deals with $\mathbb{R}^n$, in fact, mostly $\mathbb{R}^4$. The remaining chapters are dedicated to general vector spaces. I present here a brief synthesis of the results of the whole analysis.

Linear algebra notions presented within the $\mathbb{R}^2$-$\mathbb{R}^3$ model. Many notions of elementary linear algebra, and of vector spaces with an inner product, appear in the model. Yet, there are important exceptions such as vector spaces, vector subspaces, spanned subspace, and basis. (The less general notion of coordinate basis vectors is already used for spaces of dimension 1, 2 or 3.)

Use of drawings. There are 92 drawings in the book. Two of them illustrate general situations (in an arbitrary vector space), and five illustrate situations in dimension 4. The other 85 (92%) of the drawings are associated with situations in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

The elements of these vector spaces are sometimes represented as arrows, and sometimes as points. No explicit rationale is given for this, and possible confusions are not discussed.

Moving from the $\mathbb{R}^2$-$\mathbb{R}^3$ model to the general theory. The book displays two stages in generalizing from the $\mathbb{R}^2$-$\mathbb{R}^3$ model to general linear algebra.

Stage 1. Introducing $\mathbb{R}^n$ (Chapter 4). A special chapter is dedicated to generalizing from the $\mathbb{R}^2$-$\mathbb{R}^3$ model to $\mathbb{R}^n$. That chapter is very similar in structure to the previous ones, thanks to the use of coordinates. A specific choice is made by the authors; they emphasize $\mathbb{R}^4$. It appears as a first stage in the generalization, already outside of the "familiar geometry" but allowing an explicit description of vectors, thereby avoiding dots in their coordinate representations. Five drawings are given to illustrate situations in $\mathbb{R}^4$. Linear algebra in $\mathbb{R}^4$ is used as a first step towards $\mathbb{R}^n$, as an intermediate intuitive (paradigmatic) model.

Stage 2. Abstract vector spaces (Chapter 5). The introduction of abstract vector spaces (finite dimensional vector spaces over $\mathbb{R}$) is accompanied by a radical change in presentation. There are almost no drawings. Some very important notions, like vector space, vector subspace, spanned subspace and basis are introduced for the first time. The main link with the preceding chapters is $\mathbb{R}^n$, which is used as a new paradigmatic model.

No direct use is made of the $\mathbb{R}^2$-$\mathbb{R}^3$ model for abstract spaces. Rather, it is used as a paradigmatic model for $\mathbb{R}^n$; then $\mathbb{R}^n$ itself is used as a paradigmatic model for other vector spaces. The main link between the $\mathbb{R}^2$-$\mathbb{R}^3$ model and general linear algebra is that some of the terms used in the general context have already been encountered in the geometric context. However, the authors do not suggest referring to an associated drawing that might reinforce that link and help with the generalization process.

No geometric model is used for abstract vector spaces. Their introduction to abstract vector spaces does not appear as a natural generalization. Too many important notions are not included in the model.

Using the $\mathbb{R}^2$-$\mathbb{R}^3$ model as a geometric model for $\mathbb{R}^n$ seems worthwhile for shaping useful intuitions. Moreover, this can be an important stage in the generalization process. Studying the $\mathbb{R}^2$-$\mathbb{R}^3$ model can provide more general indications about possible thinking processes involved in moving from dimension 2 or 3 to dimension n, with $n > 3$. I will now examine a particular example of the way students use the $\mathbb{R}^2$-$\mathbb{R}^3$ model for $\mathbb{R}^n$.

5. Analysis of the use of the $\mathbb{R}^2$-$\mathbb{R}^3$ model by students

During the second semester of the academic year 2000–2001, I observed a six-week long linear algebra course for second-year university students. It focused on quadratic forms and vector spaces with an inner product. All the students had learned elementary linear algebra during their first year.

The course was taught by Proffesor Thomas[6], an experienced teacher and researcher at the university where the study took place. The teaching of the course consisted of two lectures ($1\frac{1}{4}$ hours each) and two tutorials (2 hours each) per week. During the lectures, all 110 of the students sat together in a large lecture hall. They copied down the lecture notes and mostly remained silent. During the tutorials, limited to groups of around 30, students attempted to solve exercises with the help of a teacher. The exercises were taken from a list given by the teacher T, who gave the lectures, one tutorial, and organized that particular teaching. After observing the class, I interviewed Professor Thomas and eight of his students individually. The student interviews were based on a questionnaire (see Appendix B) and included the following task: *"Find the length of a diagonal of a cube with edges of length 1 in $\mathbb{R}^n$."*

I present first a brief account of the teacher's interview. I discuss only the aspects that can be linked with students' solution attempts, which I analyze in the second subsection.

5.1. The Teacher's Choices.

The course was supposed to offer an overall presentation of quadratic forms, inner products, and symmetric and orthogonal matrices. The teacher introduced all these notions, but he chose to emphasize the $\mathbb{R}^2$ and $\mathbb{R}^3$ case in the sense that, after stating general results, he often illustrated them in $\mathbb{R}^2$ or $\mathbb{R}^3$. Sometimes a result was only established in $\mathbb{R}^2$ and $\mathbb{R}^3$, and the students were asked to do the generalization as homework (only for results stated with coordinates). Among the 32 exercises proposed during the corresponding tutorials, 20 were set exclusively in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Another typical choice of the teacher was the use of many drawings. He made drawings during the lectures (66 drawings during 15 hours of lecture). He also explicitly asked students to produce drawings in 10 of the 32 exercises. However, these drawings were exclusively used to illustrate situations in $\mathbb{R}^2$ or $\mathbb{R}^3$. Comparing these choices with the results of the mathematicians' questionnaire (§3) shows that they were not usual in the French teaching context.

For these reasons, I especially questioned the teacher about his use of drawings, the role of the $\mathbb{R}^2$-$\mathbb{R}^3$ model in his course, and its possible use by students. In summary, he answered these questions as follow:.

[6]This name for the teacher and the names for students are pseudonyms.

(1) *Drawings.* According to Professor Thomas, the drawings did not help understanding. They were just natural because quadratic forms were geometrical objects, stemming from physics. He said the drawings must be used only to illustrate situations in $\mathbb{R}^2$ or $\mathbb{R}^3$. Though he sometimes drew to illustrate a situation in $\mathbb{R}^n$, in such cases he asked the students to think in $\mathbb{R}^2$ or $\mathbb{R}^3$.

(2) *The $\mathbb{R}^2$-$\mathbb{R}^3$ model.* According to the teacher, the study of $\mathbb{R}^2$-$\mathbb{R}^3$ as vector spaces with the dot product as presented in his course was not intended to help with general spaces, or even with $\mathbb{R}^n$. Such study was interesting in itself and graduate students often had to manipulate general statements without being conscious of their meaning in small dimensional spaces. During the interview I insisted on a possible use of the $\mathbb{R}^2$-$\mathbb{R}^3$ model to learn, or understand, the general theory. Professor Thomas answered: "For quadratic forms, all the phenomena already happen in three-dimensional spaces. *It is necessary to understand how to move up from 2 to 3* (emphasis added). After that, there is nothing new."

The sentence emphasized above was a very important assumption of the teacher: a student who understands the underlying process when moving from $\mathbb{R}^2$ to $\mathbb{R}^3$ can easily carry out the generalization. I also asked Professor Thomas about the exercise used in student interviews and possible student answers. His assumption was: "If they are able to solve it in $\mathbb{R}^3$, they are able to solve it." The actual situation for students was more intricate.

5.2. Analysis of the students' interview responses. The interview exercise was formulated as follows:

"Find the length of a diagonal of a cube with edges of length 1 in $\mathbb{R}^n$."

One student difficulty was linked to the geometrical vocabulary. These students had never been introduced to "cubes" and "diagonals" in $\mathbb{R}^n$. This might have embarrassed some of them and prevented them from solving the exercise, even if it were possible to answer without having clear insight into the n-cube's significance.

During the interviews, I chose to intervene as little as possible in order to avoid influencing the students' solving processes. The only hint I gave was to indicate where the diagonal of a cube lies in dimension 3, if they had drawn such a cube and needed such a hint. There were two main ways to solve the problem accessible to second-year students.

Analytic solution method. The first one possible solution method was analytic. One of the diagonals of the cube can be represented as a vector[7] $\overrightarrow{OA}$, where O has coordinates $(0, 0, ..., 0)$ and A has coordinates $(1, 1, ..., 1)$. Thus the length of the diagonal is $|\overrightarrow{OA}| = \sqrt{n}$. In this case, considering the problem for $n = 2$ or $n = 3$ can be helpful; it allows one to find the coordinates of A for these values of n. That result can then be immediately generalized. The students who solved the exercise in this way were using the $\mathbb{R}^2$-$\mathbb{R}^3$ model as a paradigmatic model (in Fischbein's terminology). Then they moved up to $\mathbb{R}^n$ using coordinates. That move can be purely algebraic; but it can also be helped by the use of a figural model displaying the coordinates of A for $n = 2$ or $n = 3$.

[7] It can also be interpreted as a segment.

Inductive solution methods. A second solution is by an induction process. For $n = 2$, the length of the diagonal is $\sqrt{2}$. One assumes that the length of the $(n-1)$-cube's diagonal is $\sqrt{n-1}$. The diagonal of the n-cube is then the sum of a diagonal of the $(n-1)$-cube and of a unit vector orthogonal to it. From the Pythagorean Theorem, the length of the diagonal is thus $\sqrt{n}$ (this is the vector version; the diagonals can also be interpreted as the sides of a right triangle). A related solution method requires consideration of the case $n = 2$ and also of the process leading from $n = 2$ to $n = 3$. That process provides the key to the induction; the associated drawing could display the diagonal of a side and a unit vector orthogonal to it.

Students' solution attempts. The eight students' answers fell into three groups.

Group 1: The obstacle of geometrical vocabulary (Students: Ana, Barbara, Charles, Diane). These students did not overcome the obstacle of vocabulary. They were not able to confer any meaning to the term "cube" in $\mathbb{R}^n$.

> *Ana*: For $n = 3$, it's as cube... For n? I do not see what a cube can be! For $n = 3$, the diagonal of a face is $\sqrt{2}$; it gives $\sqrt{3}$ ... But for dimension n...

Two students did not try to use a drawing; the other two drew only the 3-cube. They all solved the problem in dimension 3 and calculated $\sqrt{3}$. Barbara also mentioned a square as a cube in dimension 2 and calculated $\sqrt{2}$ in that case. But she did not link the two cases (dimension 2 and 3); she did not identify a generalizable process.

Group 2: Using coordinates (Student: Edouard). There was only one student in Group 2. I give here details about his reasoning process because it was very different from that of all the others. He said at the beginning: "I first do it in dimension 2." Then he made a drawing (Figure 2). He calculated the value $\sqrt{2}$ immediately after plotting the diagonal and wrote that on his drawing.

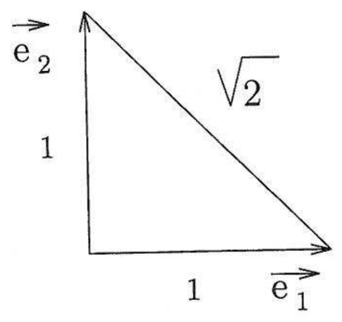


FIGURE 2. Student Edouard. Diagonal of the cube in $\mathbb{R}^2$.

He went on, speaking and drawing simultaneously:

> *Edouard*: I do the same now in dimension 3. e_1, e_2, e_3 is an orthonormal basis of $\mathbb{R}^3$. The diagonal of the cube is $\overrightarrow{OA}$, A is there... So the coordinates of A are all equal to 1, it gives $\sqrt{3}$... And it will always be the same thing, the coordinates of A are 1, 1, 1 ... So $|\overrightarrow{OA}| = \sqrt{n}$.

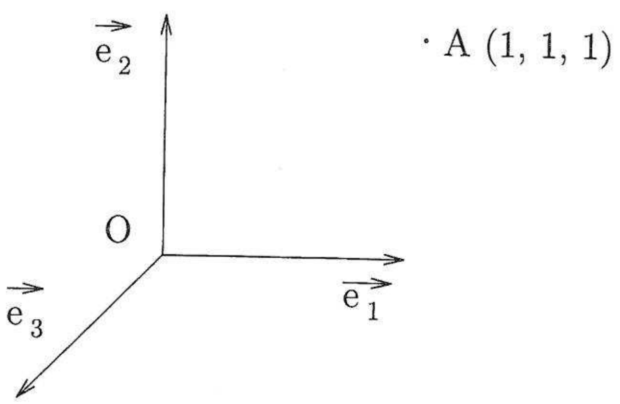


FIGURE 3. Student Edouard. Diagonal of the cube in $\mathbb{R}^3$.

The corresponding drawing is shown in Figure 3.

Edouard was the only student who produced a drawing with vectors. He was also the only student who made a drawing but did not draw a complete 3-cube. On his first drawing, he plotted the diagonal; on the second, only its endpoints. He immediately thought in terms of vectors and arrows. Then he recognized an orthonormal basis on his first drawing, and was thus led to use coordinates. He identified a linear algebra context; then instead of using drawings stemming from secondary school geometry like the others, he used a figural model associated with linear algebra. It helped him answer the question for $n = 2$ and $n = 3$, and to formulate the problem with coordinates. Then the move from the $\mathbb{R}^2$-$\mathbb{R}^3$ model to the general case became obvious to him.

Group 3: Induction process (Students: Fanny, Guy, Henri). These three students first drew a square and then calculated $\sqrt{2}$ for $n = 2$; then they drew a cube and used the Pythagorean Theorem, explicitly or not, to compute $\sqrt{3}$ for $n = 3$. They all claimed that the general result was $\sqrt{n}$. They all used a kind of induction process, but none of them produced a rigorous proof. The most algebraic reasoning was produced by Fanny, who proposed no geometrical interpretation or justification for its generalization:

> *Fanny*: For $n = 2$, it gives $\sqrt{a^2 + a^2} = a\sqrt{2}$. Then for $n = 3$, I have $\sqrt{a^2 + 2a^2} = a\sqrt{3}$. Then it will go on the same way, there is always another a^2, and you get $\sqrt{a^2 + (n-1)a^2} = a\sqrt{n}$.

(rather than labeling them with 1, she labeled the edges a). Even if the calculations were not very different from those produced by thinking in terms of coordinates, the reasoning process was not the same. She was first dealing with lengths of segments and then with algebraic expressions, without any geometrical interpretation.

By contrast, the reasoning of Henri was based on geometrical statements. After calculating $\sqrt{2}$ for $n = 2$ and using it to deduce $\sqrt{3}$ for $n = 3$, he said:

> *Henri*: I would say then there are 1, and $\sqrt{3}$, and it is orthogonal, so it gives $\sqrt{4} = 2$, but I'm not sure, I do not see it clearly... But I think it works, because $\sqrt{3}$ is the diagonal, and the last edge is

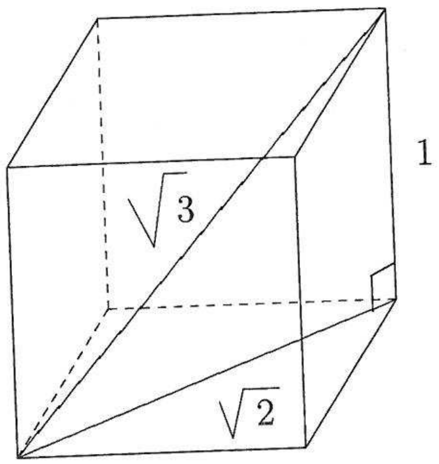


FIGURE 4. Student Henri. The diagonal of the cube in $\mathbb{R}^3$.

orthogonal to it. So it is actually 2 for $n = 4$. And it is always the same, so you obtain $\sqrt{\sqrt{\sqrt{1+1}^2 + ... + 1}^2 + 1} = \sqrt{n}$.

At first he appeared to feel embarrassed because he lacked a picture depicting the n-cube. But his drawing of the 3-cube displayed a right triangle formed by the diagonal of a face, an edge of the cube, and the corresponding diagonal of the cube (as in Figure 4).

After producing this drawing, Henri focused on the orthogonality of the diagonal of a face and an edge of the cube. Even if he did not explicitly interpret the diagonal of the 3-cube as the diagonal of a "face" of the 4-cube, he used it that way; this allowed him to compute the result for $n = 4$. He did not try to provide any further geometrical interpretation, but immediately generalized his result for any value of n.

The observations described in this section came from a clinical study about a specific task, conducted with a small number of students. However, the behaviors described illustrate the more general phenomena discussed below.

5.3. Conclusions. The first issue I address here is the use of drawings. They played a central part in the interview exercise. The two students who did not make any drawing did not produce a solution. The one student who used vectors was immediately led to the solution. Representing the whole cube did not help. But the four students who represented a square, then a 3-cube, and interpreted the faces of the 3-cube as squares (one of them even drew the 3-cube *on* the square), found the solution.

5.3.1. *Using Coordinates.* Recall that the teacher said: "If they can do it in $\mathbb{R}^3$, they can do it." The interviews with students invalidated that claim. The teacher may have thought that the students were going to solve the exercise analytically, but only one student did so. The word "cube" placed all the other students in a geometrical context where they stayed captured. Moreover, during their tutorials, the students had never encountered tasks that requested analytic solutions where no coordinates were given in the task statement.

Using coordinates can be a good way to move from dimension 2 or 3 to dimension n. But students may not use such a process in their reasoning if coordinates are not a familiar tool for them. In particular, they must be able to formulate problems presented in the abstract, or geometric, mode in terms of coordinates (Hillel, 2000). Fostering this ability takes a specially designed course.

5.3.2. *Increasing the dimension.* The teacher also made a more general claim about quadratic forms and inner product spaces: "The important point is to be able to move from dimension 2 to dimension 3."[8] The observations in the particular interview exercise studied here are consistent with that claim. The four students who did not identify the process that led from $n = 2$ to $n = 3$ did not solve the exercise. The four others found that the length of the diagonal was $\sqrt{n}$.

Moving up from dimension $n-1$ to dimension n is a complex process. In particular, it is necessary to interpret the space of dimension $n-1$ as a hyperplane of the space of dimension n. Mathematicians are quite used to such processes; they readily consider the first space as an hyperplane and then add a supplementary line to it, to obtain the whole n-space. This is a difficult process for students and requires familiarity with the notion of subspaces. Such a supplementing process could be explicitly addressed in linear algebra tutorials. The $\mathbb{R}^2$-$\mathbb{R}^3$ model could be used to study how to move from dimension 2 to dimension 3.

5.3.3. *Decreasing the dimension.* For the task discussed here, it was sufficient at each stage of the induction to reason in a plane that contained the $(n-1)$-diagonal and the n-diagonal. This is a familiar process for mathematicians. Many general results can be established by reasoning in a well-chosen 2-dimensional space, and it is then possible to help the reasoning with a drawing. The actual n-dimensional object cannot be pictured, but it is always possible to cut it along a plane and represent the obtained section. Henri was led to a solution by the use of such a process in $\mathbb{R}^4$. That possibility (cutting and drawing) could be explicitly emphasized in a linear algebra course. In that case, there would be no intervention of a geometric model. The figural model would be directly associated with the pertinent part of the general situation.

6. Conclusion

In Sections 3, 4, and 5, I reported on different aspects of my work: a mathematicians' questionnaire, results from a textbook study, and interviews with teacher and students. Some of the results obtained are general statements about the use of geometric or figural models in linear algebra; others are relative to the use of the $\mathbb{R}^2$-$\mathbb{R}^3$ model. I will now synthesize the answers provided by these approaches under each of the research questions presented in Section 2.

6.1. What are the possible uses of geometric models in linear algebra? The first answer is: it appears that linear algebra cannot be taught nor learned as a mere generalization of a geometry. The historical development of linear algebra (Dorier, 2000) has indicated that the modern theory emerged from the necessity of unification of several mathematical domains. The intellectual need for linear algebra (I refer here to Harel's (2000) Necessity Principle) is grounded in the unification of several mathematical domains.

[8]Banchoff and Wermer make a similar claim in their book. It can be formulated as, "If students can move up from $\mathbb{R}^3$ to $\mathbb{R}^4$, then they will have no problem with $\mathbb{R}^n$."

Yet, a geometric model can be helpful, especially because the associated figural model confers on the geometric model the appearance of concreteness. For example, the $\mathbb{R}^2$-$\mathbb{R}^3$ model allows one to present some notions and results of linear algebra before introducing the general theory as some properties appear as self-evident in a geometric context. The use of coordinates allows one to move up to $\mathbb{R}^n$. However, there is no evidence that this model can be used effectively to introduce abstract vector spaces.

6.2. How do mathematicians and students use geometrical and figural models in linear algebra? Most of the mathematicians in France advocate one of two opposite approaches. The first group advocates a structural approach to linear algebra, without geometrical or figural models. The second group recommends a geometry course before linear algebra so that geometry can then provide models and the associated drawings. In both cases, however, the mathematicians do not develop a figural model specifically for linear algebra; their drawings are only used in a geometrical context.

I did not discuss in this paper the general use of geometric models by students. Yet, the importance of familiarity with models must be emphasized. It was a direct consequence of the Necessity Principle (Harel, 2000) and was also observed by Sierpinska (2000). Students may use familiar models in their reasoning processes, even when a teacher proposes a geometric model for linear algebra. These familiar models can be inappropriate for linear algebra.

6.3. What are the consequences of the observed uses of models on students' practices and thinking processes? I studied this question in a particular context: the use of the $\mathbb{R}^2$-$\mathbb{R}^3$ model by students in their solving of a problem stated in geometric language in $\mathbb{R}^n$. The model can help students find an algebraic description of the problem that can be generalized to higher dimensions, provided they use coordinates. That possibility is strongly linked with the existence of an appropriate figural model that allows one to derive an algebraic description.

Understanding the process leading from $\mathbb{R}^2$ to $\mathbb{R}^3$ provides another possibility, if the student extends that process so as to use it in going from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n$. The figural model is fundamental in that case as well, to help in understanding the generalization mechanism. But the model can also have negative effects for some students who stay captured in the geometrical context.

These results indicate that geometric models must be used carefully in linear algebra courses. Geometry cannot be the only starting point for linear algebra; other domains must intervene to justify the need for a general theory.

The geometric model requires long-term teaching so that it will become very familiar to students. A figural model, specially intended for linear algebra, should be presented. However, the uses of such a model for general vector spaces requires additional research.

Moreover, because geometric models belong to dimension 2 or 3, it might be useful to include in a linear algebra course the study of processes used by experts as they move from dimension n to dimension $n+1$. And to discuss how they recognize, in an n-dimensional problem, that the main phenomenon occurs in a well-chosen 2 or 3-dimensional space. Linear algebra teaching could integrate these possibilities in order to try explicitly to develop students' geometric intuition. For the moment, this does not seem to be done by mathematicians, at least in France.

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Appendix A: Mathematicians' Questionnaire

1. Use of geometry in the exercises.

Question 1.1: The following exercise is often proposed to second year students.

Let E be an inner product space, and p a projection of E. Prove that p is an orthogonal projection if and only if, for all x in E, $|x| \geq |p(x)|$.

- What solution would you present to the students?
- Are there geometrical aspects in the solution you propose, and for what purpose do you use them?
- Would you use a drawing in a solution presented to the students? If the answer is positive, which drawing would you use, and what do you expect from the use of that drawing?

Question 1.2: The following exercise is often proposed to first year students.

Let E be a vector space, and x, y, z three vectors in E, linearly independent by pairs. Is the set of the three vectors (x, y, z) linearly independent?

- If you observe during a tutorial a student who says that (s)he is sure that the answer is positive, but (s)he can not find a proof, what do you tell him (her) to help? (Give a precise answer, and explain the reason for your choice).
- The same exercise is proposed in an examination. A student proposes the following solution:

  No, the vectors drawn hereby provide a counter-example.

  With the drawing of Figure 5.

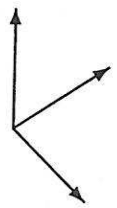


FIGURE 5

  - What mark, between 0 and 5, do you attribute to this answer?
  - Which comments do you write on the student's sheet?
  - Explain your mark and comments.

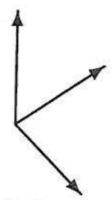
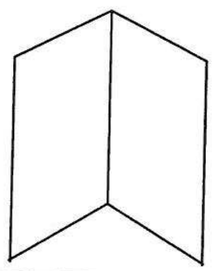
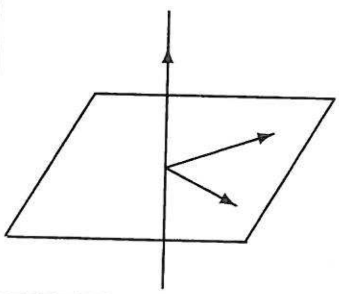
2. Use of the geometry taught in secondary school.

Question 2.1: In secondary school, the students encounter the words "basis" and "orthonormal basis" in the geometry courses. Do you think that some of the properties, techniques, results...presented in secondary school can be used at university in the linear algebra courses? If your answer is negative, explain why. If your answer is positive, present the results you consider useful and explain how they can be used.

Question 2.2: In secondary school, the students encounter the words "projection" and "orthogonal projection" in the geometry courses. Do you think that some of the properties, techniques, results...presented in secondary school can be used at university in the linear algebra courses? If your answer is negative, explain why. If your answer is positive, present the results you consider useful and explain how they can be used.

3. Use of drawings in linear algebra

Question 3.1: For each of the drawings in the table, indicate if you use it in your linear algebra courses; if the answer is "yes" indicate which notions or properties you illustrate with it. (You can mention several uses of the same drawing.)

Drawing	Used (yes/no)	That drawing illustrates
		
		
		

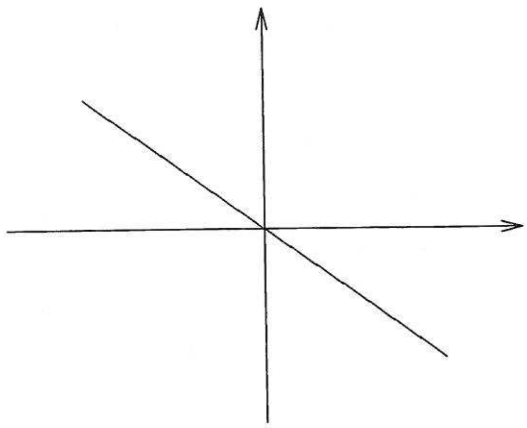
Question 3.2: If you use other drawings, draw them in the following table, and indicate the interpretation(s) you associate with them. (A blank table with five lines followed.)

Appendix B: Students' Questionnaire

Drawings on the exam This section refers to the text of an exam passed by the students two weeks before the interviews. I also used their own exam sheets. Here are the two exercises of the exam that I used in the interviews.

> **Exercise 1 of the exam**
> *Let q be the quadratic form defined by:* $q(x) = (2x_1 + 3x_2)^2$. *Give its matrix, its rank, and draw its isotropic cone.*
> **Exercise 2 of the exam**
> *Let q be the quadratic form defined on $\mathbb{R}^3$ by:*
> $q(x) = 3x_1^2 + 2x_2^2 - x_3^2$. *Give its rank, and draw its isotropic cone. Is q positive ?*

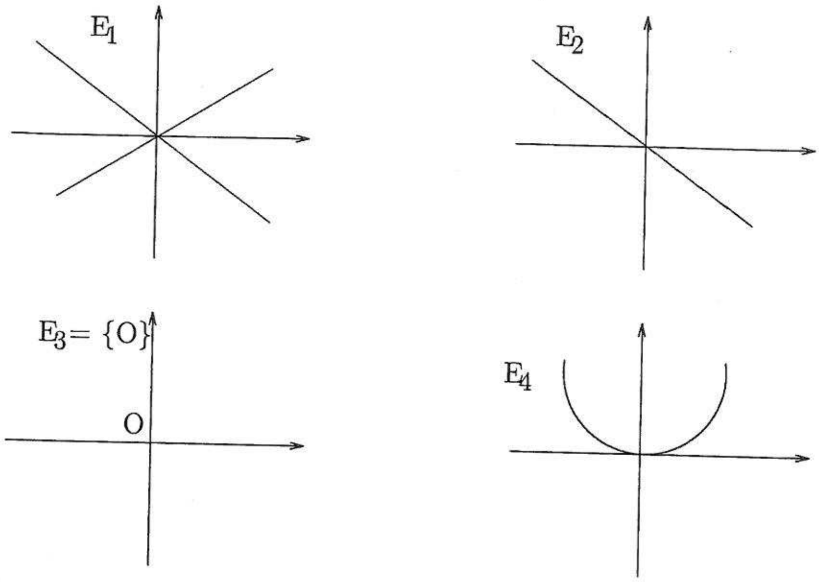
About Exercise 1 of the exam, I asked the following question:
The isotropic cone is a straight line that can be illustrated by the following figure. How would you represent an element of that cone?



About Exercise 2 of the exam, I asked the following questions:
If the student drew axes, does he or she think that these axes were part of the requirement? Was the drawing useful to answer the question: *Is q positive*?

Other Exercises

> **Exercise 1**
> *Each of the drawings in the figure represents a subset E_i of $\mathbb{R}^2$. In each case, indicate if it is possible to find a quadratic form q of $\mathbb{R}^2$ such that:*
> – *E_i is the kernel of q?*
> – *E_i is the isotropic cone of q?*



Exercise 2
Let $E = \mathbb{R}_3[X]$ be the inner product space of degree 3 polynomials and let P and Q be two orthogonal elements of E whose length is 1. Can you determine the length of $P + Q$?

Exercise 3
Find the length of a diagonal of a cube with edges of length 1 in $\mathbb{R}^n$.

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